UNKNOTTING RECTANGULAR DIAGRAMS OF THE TRIVIAL KNOT BY EXCHANGING MOVES

CHUICHIRO HAYASHI AND SAYAKA YAMADA

Abstract. If a rectangular diagram represents the trivial knot, then it can be deformed into the rectangular diagram with only two vertical edges by a finite sequence of merge operations and exchange operations, without increasing the number of vertical edges, which was shown by I. A. Dynnikov. We show in this paper that we need no merge operations to deform a rectangular diagram of the trivial knot to one with no crossings.

1. Introduction

Birman and Menasco introduced arc-presentation of knots and links in [1], and Cromwell formulated it in [2]. Dynnikov pointed out in [3] and [4] that Cromwell's argument in [2] almost shows that any arc-presentation of a split link can be deformed into one which is "visibly split" by a finite sequence of exchange moves. He also showed that any arc-presentation of the trivial knot can be deformed into one with only two arcs by a finite sequence of merge moves and exchange moves, without using divide moves which increase the number of arcs. As is shown in page 41 in [2], an arc-presentation is equivalent to a rectangular diagram. All arguments in this paper are described in words on rectangular diagrams.

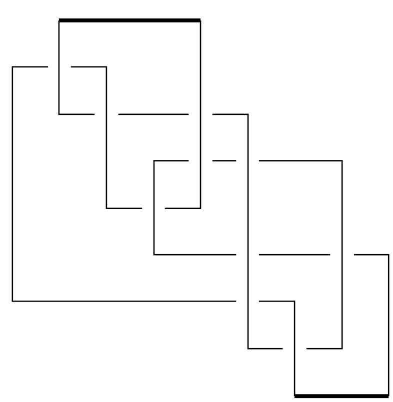

Figure 1. rectangular diagram

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A *rectangular diagram* of a knot is a knot diagram in the plane $\mathbb{R}^2$ which is composed of vertical lines and horizontal lines such that no pair of vertical lines are colinear, no pair of horizontal lines are colinear, and the vertical line passes over the horizontal line at each crossing. See Figure 1. These vertical lines and horizontal lines are called *edges* of the rectangular diagram. Every knot or link has a rectangular diagram (Proposition in page 42 in [2]).

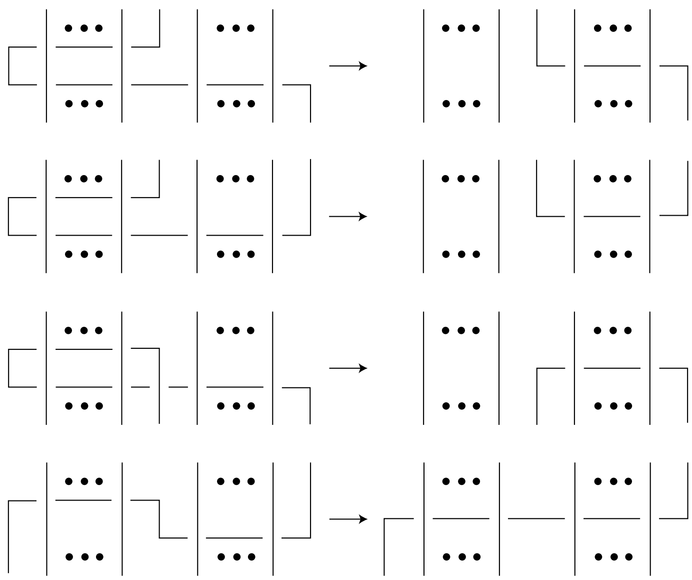

FIGURE 2. interior horizontal merge

Cromwell moves, which are described in the next three paragraphs, are elementary moves for rectangular diagrams of knots and links. They do not change type of knots and links. Moreover, Theorem in page 45 in [2] and Proposition 4 in [3] state that, if two rectangular diagrams represent the same knot or link, then one is obtained from the other by a finite sequence of these elementary moves and rotation moves, which is introduced below.

First, we recall merge moves. If two horizontal (resp. vertical) edges connected by a single vertical (resp. horizontal) edge have no other horizontal (resp. vertical) edges between their abscissae (resp. ordinates), then we can amalgamate the three edges into a single horizontal (resp. vertical) edge. This move is called an *interior horizontal (resp. vertical) merge*. See Figure 2 for examples of interior horizontal merge moves. If two horizontal (resp. vertical) edges connected by a single vertical (resp. horizontal) edge have the other horizontal (resp. vertical) edges between their abscissae (resp. ordinates), i.e., they are the top and bottom (the leftmost and rightmost) edges, then we can amalgamate the three edges into a single horizontal (resp. vertical) edge. We may place the new horizontal (resp. vertical) edge either at the top height or at the bottom height (resp. either in the leftmost position or in the rightmost position). See Figure 3. We call this move an *exterior horizontal (resp.*

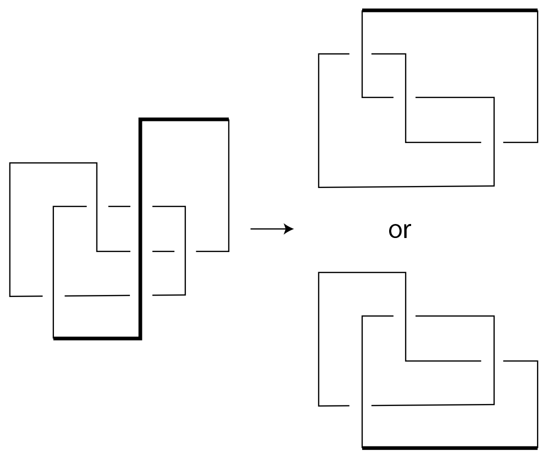

FIGURE 3. exterior horizontal merge

vertical) merge. Note that a merge move decreases the number of vertical edges and that of horizontal edges by one. The inverse moves of merge moves are called *divide moves*.

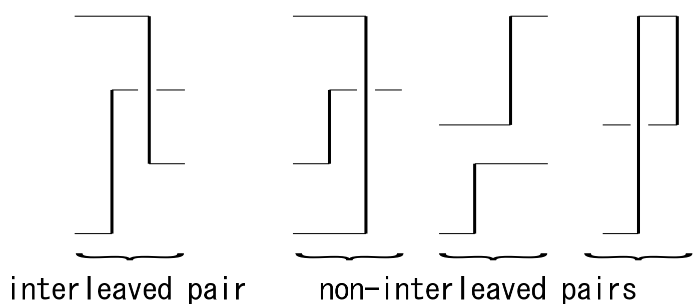

FIGURE 4. interleaved pair and non-interleaved pairs

To describe exchange moves, we need a terminology. Two vertical edges are said to be *interleaved*, if the heights of their endpoints alternate. See Figure 4. Similarly, we define interleaved two horizontal edges.

If two horizontal edges at mutually adjacent heights are not interleaved, then we can exchange their heights. See Figure 5. This move is called an *interior horizontal exchange*. If the top horizontal edge and the bottom one are not interleaved, then we can exchange their heights. We call this move an *exterior horizontal exchange*. See Figure 6, where the rectangular diagram obtained from one in Figure 1 by an exterior horizontal exchange move. Similarly, we define *vertical exchange* moves.

A *horizontal rotation move* on a rectangular diagram moves the top edge to the bottom, or the bottom edge to the top. A *vertical rotation move* on a rectangular diagram moves the leftmost edge to the right, or the rightmost edges to the left. (The horizontal (resp. vertical) rotation move corresponds to the $\pm 2\pi/n$ rotation about the dual axis (resp. the axis) on an arc-presentation.) However, we do not use rotation move in this paper.

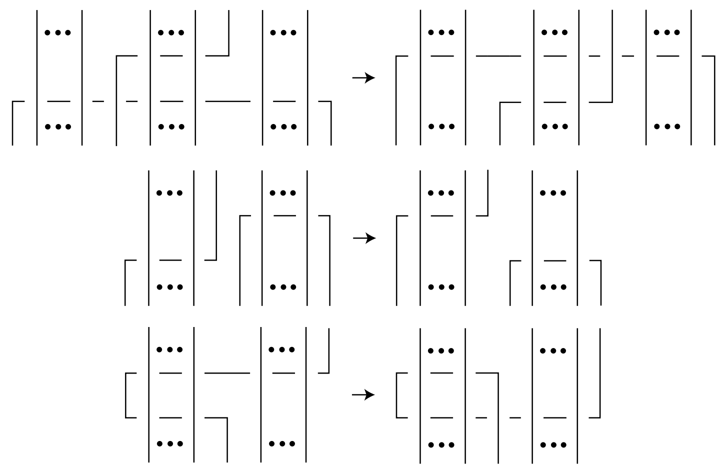

FIGURE 5. interior horizontal exchange

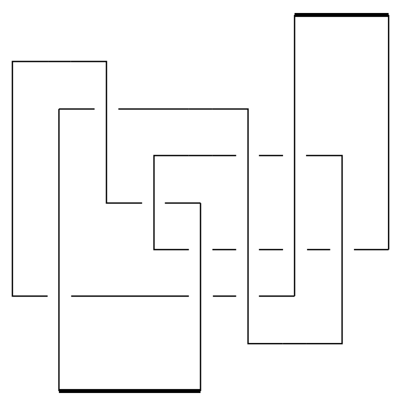

FIGURE 6. This is obtained from Figure 1 by an exterior horizontal exchange.

The next result of Dynnikov gives a finite algorithm to decide whether a given rectangular diagram represents the trivial knot or not.

Theorem 1.1 (Dynnikov [3], [4]). *Any rectangle diagram of the trivial knot can be deformed into one with only two vertical edges by a finite sequence of merge moves and exchange moves.*

Note that the sequence in the above theorem contains no divide moves. Hence the sequence gives a monotone simplification, that is, no move in the sequence increases the number of vertical edges. There are only finitely many rectangle diagrams with a fixed number of vertical edges. Thus the above theorem gives a finite algorithm for the decision problem.

The sequence as in Dynnikov's theorem sometimes needs to contain exterior exchange moves. In fact, the rectangle diagram shown in Figure 1 represents the trivial knot. It admits no merge moves since it does not have an edge of length 1 or $9-1$. We cannot

apply any interior horizontal exchange move to the diagram because every pair of horizontal edges in adjacent levels are interleaved. It can be seen easily that no vetical exchange move can be performed on this diagram. Hence every sequence as in Dynnikov's theorem on this diagram must begin with the exetrior horizontal exchange move.

In [5], A. Henrich and L. Kauffman announced an upper bound of the number of Reidemeister moves needed for unknotting by applying Dynnikov's theorem to rectangular diagrams. Lemma 7 in [5] states that no more than $n-2$ Reidemeister moves are required to perform an exchange move on a rectangular diagram with n vertical edges. However, their proof of Lemma 7 in [5] does not consider the exterior exchange moves.

In this paper, we show that the decision problem can be solved without using merge moves. In fact, we obtain the next result by taking advantage of Dynnikov's theorem.

Theorem 1.2. *For any pair of rectangle diagrams of the trivial knot with the same number of vertical edges, there is a finite sequence of exchange moves which deform one into the other.*

Corollary 1.3. *Any rectangle diagram of the trivial knot can be deformed into one with no crossings by a finite sequence of exchange moves.*

We need much larger number of Cromwell moves in a sequence as in this corollary than in that in Dynnikov's theorem. However, this improves Lemma 6 in [5] which assures that the number of exchange moves and merge moves in a sequence as in Dynnikov's theorem is bounded above by $\sum_{i=2}^{n} \frac{1}{2} i[(i-1)!]^2$, where n is the number of vertical edges of the first rectangular diagram. In fact, we obtain the upper bound $\frac{1}{2}n[(n-1)!]^2$ which is an upper bound for the number of combinatorially distinct rectaugluar diagrams with n vertical edges given in Proposition 5 in [5]

2. Proof of Theorem 1.2

We prove Theorem 1.2 in this section.

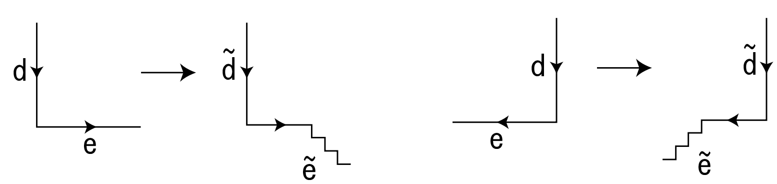

FIGURE 7. nothced edge

Let R, $\tilde{R}$ be oriented rectangular diagrams of the same knot. We say that R is an *outline* of $\tilde{R}$ if $\tilde{R}$ is obtained from R by replacing the edges of R by *notched edges* as below. Let e be an original edge of R, and d the edge preceding e. Then the notched edge $\tilde{e}$ substituting for e consists of a long edge followed by 0 or even number of very short edges going away

from the right angle between d and e as in Figure 7. We say that a notched edge has r notches if it consists of $2r+1$ edges. In Figure 7, $\tilde{d}$ has no notches for simplicity. See Figure 8 for entire view of R and $\tilde{R}$. Note that $\tilde{R}$ has the same number of crossings as R.

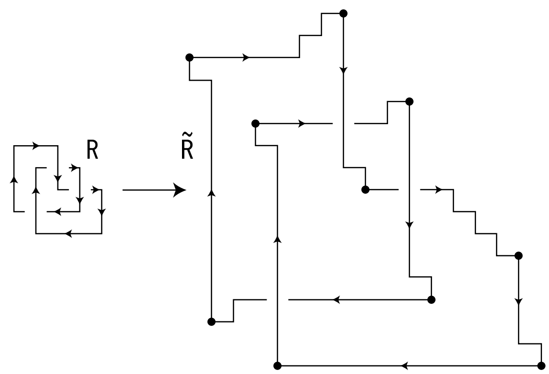

FIGURE 8. outline

Lemma 2.1. *Let R be an oriented rectangular diagram of a knot, and $\tilde{R}$ an oriented rectangular diagram which has R as its outline. Then for each edge e of R, there is an oriented rectangular diagram $\tilde{R}_e$ such that (1) $\tilde{R}_e$ is obtained from $\tilde{R}$ by a finite sequence of exchange moves, that (2) R is an outline of $\tilde{R}_e$ and that (3) $\tilde{R}$ has only one notched edge $\tilde{e}$ with positive number of notches and $\tilde{e}$ substitutes for e.*

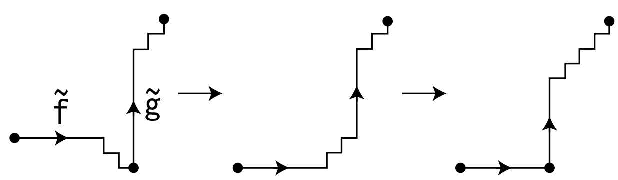

FIGURE 9. moving notches

Proof. Let S be an oriented rectangular diagram of a knot, and $\tilde{S}$ an oriented rectangular diagram which has S as its outline. Let f be an edge of R, g the edge of R next to f, $\tilde{f}$ and $\tilde{g}$ notched edges of $\tilde{S}$ corresponding to f and g, r_f and r_g the number of notches of $\tilde{f}$ and $\tilde{g}$ respectively. We show that an adequate finite sequence of exchange moves brings the r_f notches of $\tilde{f}$ to behind the tail of the long edge of $\tilde{g}$.

Assume, without loss of generality, that f is horizontal and g is vertical. Let c_g be the number of horizontal edges which cross g. If the notches of $\tilde{f}$ goes into the right angle between f and g, then we simply exchange horizontal edges, say h_f, of notches of f and c_g horizontal notched edges crossing $\tilde{g}$. Precisely, we need more exchange moves if there are horizontal edges away from $\tilde{g}$ between the height of the top of h_f and the height of the

bottom of notches of $\tilde{g}$. If the notches of $\tilde{f}$ goes away from the right angle between f and g, then we first exchange every pair of $r_f + 1$ horizontal edges of $\tilde{f}$ to obtain the situation as in the previous case. See Figure 9. Note that edges of notches are not interleaved with any edge since they are very short.

Then, by applying operations as above to R repeatedly, we obtain the lemma. □

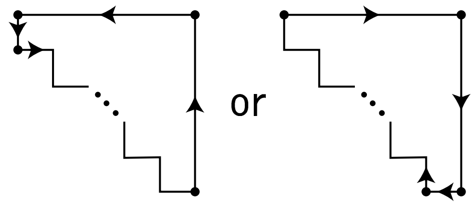

FIGURE 10. the rectangular diagram Δ with Φ being an outline

Let R_0 be a rectangular diagram with n vertical edges which represents the trivial knot. We give R_0 an arbitrary orientation. By Dynnikov's theorem, there is a finite sequence of merge moves and exchange moves which deforms R_0 to the rectangle diagram Φ with only two vertical edges, and performs a merge move whenever there are applicable merge moves. We take such a sequence, fix it, and call it a *Dynnikov sequence.* Let $R_0 \to R_1 \to R_2 \to \cdots \to R_m = \Phi$ be the sequence of the rectangular diagrams such that R_i is obtained from R_{i-1} by the i th move in the Dynnikov sequence. We set $\tilde{R}_0 = R_0$, and will form a rectangle diagram $\tilde{R}_i$ with R_i being an outline so that $\tilde{R}_{i+1}$ is obtained from $\tilde{R}_i$ by a finite sequence of exchange moves. Note that $\tilde{R}_m$ has no crossings because R_m has only two vertical edges. Moreover, $\tilde{R}_m$ can be deformed into one of the two diagrams in Figure 10 by Lemma 2.1. Note that, if we ignore their orientations, these two rectangular diagrams are the same, which we denote by Δ. This implies Theorem 1.2. Let T_1 and T_2 be rectangular diagrams representing the trivial knot. For $i = 1$ and 2, there is a sequence γ_i of merge and exchange moves which deforms T_i into Δ. Then the sequence γ_1 followed by the inverse of γ_2 deforms T_1 into T_2. Hence, it is enough to show the next lemma.

Lemma 2.2. *Let $S_0 \to S_1 \to S_2 \to \cdots \to S_m$ be a sequence of rectangular diagrams of a knot such that S_i is obtained from S_{i-1} by a merge or exchange move. Then there is another sequence $\tilde{S}_0 \to \tilde{S}_1 \to \tilde{S}_2 \to \cdots \to \tilde{S}_m$ with $\tilde{S}_0 = S_0$ such that S_i is an outline of $\tilde{S}_i$ and that $\tilde{S}_i$ is obtained from $\tilde{S}_{i-1}$ by a finite sequence of exchange moves.*

In the above lemma, note that the knot may not be trivial. We show the way how to construct $\tilde{S}_i$ from $\tilde{S}_{i-1}$ refering to the i th move $S_{i-1} \to S_i$ in the original sequence.

First, we apply Lemma 2.1 to $\tilde{S}_{i-1}$ so that the notches are gathered to a single notched edge which is away from the notched edges corresponding to the edges participating in the i th move on S_{i-1}.

When the i th move is an exchange move, we simply exchange the corresponding notched edges with no notches.

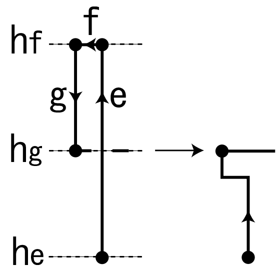

FIGURE 11. a sequence of exchange moves substituting for an interior vertcial merge move

In the case where the i th move is a merge move, let e, f and g be edges of S_{i-1} such that they appear in succession in this order in the knot and that the i th move merges these three edges into a single edge. Assume, without loss of generality, that e and g are vertical and f is horizontal. Let h_f be the height of f, and h_e and h_g the heights of endpoints of e and g which are not equal to h_f. The notched edges of $\tilde{S}_{i-1}$ corresponding to e, f, g have no notches and are called e, f, g for simplicity in the followings.

First, suppose that the i th move is an interior merge move. Then we move f in $\tilde{S}_{i-1}$ to a height h adjacent to h_g by exchange moves. When $h_g < h_e$ (resp. $h_e < h_g$), we take h to be $h_g + \epsilon$ (resp. $h_g - \epsilon$), where ϵ is a very small positive real number. See Figure 11.

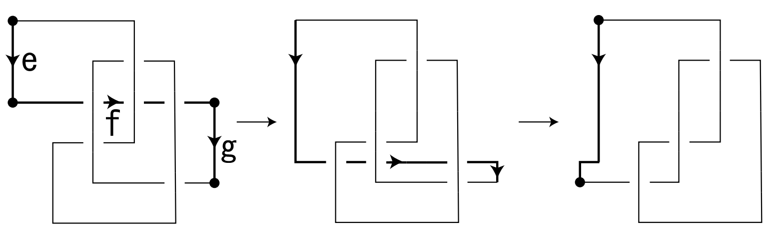

FIGURE 12. a sequence of exchange moves substituting for an exterior vertical merge move

When the i th move is an exterior merge move, let k be the edge of S_i obtained by amalgamating the edges e, f, g. First, we consider the case where the new edge k is in the vertical line of the same lateral coordinate as e. We move f in $\tilde{S}_{i-1}$ by exchange moves to the same height as in the previous paragraph, to make g very short. Then we move g by exchange moves to the position very close to e so that it forms an edge of a notch. In the case where k is in the vertical line of the same lateral coordinate as g, we move f in $\tilde{S}_{i-1}$ by exchange moves to the height $h_e + \epsilon$ (resp. $h_e - \epsilon$) to make e very short when $h_e < h_g$ (resp. $h_e > h_g$). Then we move shortened e by exchange move to the position slightly before g. Then we move the horizontal edge between e and g by exchange moves to the height very

close to h_g so that it forms an edge of a notch. See Figure 13. This completes the proof of Lemma 2.2, and hence of Theorem 1.2.

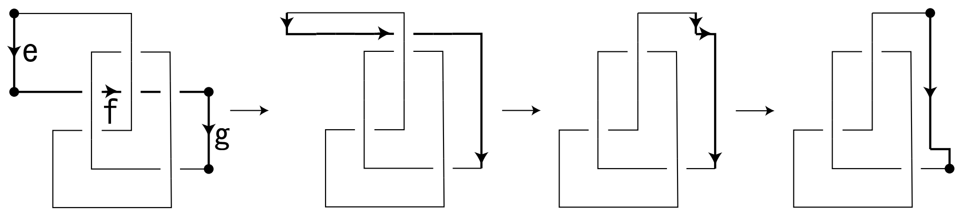

FIGURE 13. a sequence of exchamge moves substituting for an exterior vertical merge move

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